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A NEW SOLUTION OF THE GENERALIZED PROBLEM OF THE MOTION OF A BODY WITH A FIXED POINT†

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The survey [1] of results obtained in the study of the conditions for the existence of motion with precession in the dynamics of systems of rigid bodies indicates that in the classical problem concerned with the motion of a rigid body precession of general form occurs [2], the conditions for the existence of which have not been studied in the generalized problem. The present paper fills this gap. A new solution of the generalized problem characterized by the presence of precession of general form is constructed. The case of integrability, which was presented in [2] for the classical problem, is a special case of this solution.

1. FORMULATION OF THE PROBLEM

CONSIDER the equations of motion of a rigid body with a fixed point in the generalized problem

$$A\boldsymbol{\omega}^{\boldsymbol{\cdot}}=(A\boldsymbol{\omega}+\boldsymbol{\lambda})\times\boldsymbol{\omega}+\boldsymbol{\omega}\times B\mathbf{v}+\mathbf{s}\times\mathbf{v}+\mathbf{v}\times C\mathbf{v}$$
$$\mathbf{v}^{\boldsymbol{\cdot}}=\mathbf{v}\times\boldsymbol{\omega} \tag{1.1}$$

These equations have the first integrals

$$A\boldsymbol{\omega}\cdot\boldsymbol{\omega}-2(\mathbf{s}\cdot\mathbf{v})+C\mathbf{v}\cdot\mathbf{v}=2E,\quad \mathbf{v}\cdot\mathbf{v}=1$$
$$2(A\boldsymbol{\omega}+\boldsymbol{\lambda})\cdot\mathbf{v}-B\mathbf{v}\cdot\mathbf{v}=2k \tag{1.2}$$

Let **i**, **j**, **k** be the unit vectors of a system of coordinates attached to the body. Then, using the notation $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}=(a_1, a_2, a_3)$ for any vector, we have the angular velocity vector $\boldsymbol{\omega}=(\omega_1, \omega_2, \omega_3)$ of the body, the unit vector of the vertical axis $\boldsymbol{\nu}=(\nu_1, \nu_2, \nu_3)$, the gyrostatic moment $\boldsymbol{\lambda}=(\lambda_1, \lambda_2, \lambda_3)$, the generalized centre of mass vector $\mathbf{s}=(s_1, s_2, s_3)$, and $A\boldsymbol{\omega}=(x_1, x_2, x_3)$, where

$$x_1=A_{11}\omega_1+A_{12}\omega_2+A_{13}\omega_3,\quad x_2=A_{12}\omega_1+A_{22}\omega_2+A_{23}\omega_3 \tag{1.3}$$
$$x_3=A_{13}\omega_1+A_{23}\omega_2+A_{33}\omega_3$$

are the components of the angular momentum, $B\boldsymbol{\nu}$ and $C\boldsymbol{\nu}$ being vectors of the following form:

$$B\mathbf{v}=(B_{11}\nu_1+B_{12}\nu_2+B_{13}\nu_3,\ B_{12}\nu_1+B_{22}\nu_2+B_{23}\nu_3,\ B_{13}\nu_1+B_{23}\nu_2+B_{33}\nu_3)$$
$$C\mathbf{v}=(C_{11}\nu_1+C_{12}\nu_2+C_{13}\nu_3,\ C_{12}\nu_1+C_{22}\nu_2+C_{23}\nu_3,\ C_{13}\nu_1+C_{23}\nu_2+C_{33}\nu_3) \tag{1.4}$$

Thus the matrices A, B and C in (1.1) and (1.2) are symmetrical and, in addition, A is positive definite.

We say that the body undergoes a motion with precession about the vertical axis if the angle between **a** and $\boldsymbol{\nu}$, **a** being a unit vector fixed inside the body ($\mathbf{a}^{\boldsymbol{\cdot}}=0$), remains constant for the whole duration of the motion. Such a motion is characterized by the obvious invariant relation

$$\mathbf{a}\cdot\mathbf{v}=a_0,\quad a_0=\cos\theta_0 \tag{1.5}$$

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where θ_0 is the angle between $\mathbf{a}$ and $\boldsymbol{\nu}$. Differentiating (1.5), we get $\mathbf{a}\cdot(\boldsymbol{\nu}\times\boldsymbol{\omega})=0$ by virtue of the second equation in (1.1). It follows that $\boldsymbol{\omega}$ can be represented in the form

$$\boldsymbol{\omega}=\varphi^{\bullet}\mathbf{a}+\psi^{\bullet}\mathbf{v} \tag{1.6}$$

We will not consider the case when $\mathbf{a}$ and $\boldsymbol{\nu}$ are collinear, since it leads to uniform motion of the body. On substituting (1.6) into the second equation in (1.1), we have

$$\mathbf{v}^{\bullet}=\varphi^{\bullet}(\mathbf{v}\times\mathbf{a}) \tag{1.7}$$

In (1.6) and (1.7) $\varphi^{\bullet}$ and $\psi^{\bullet}$ are functions of time. A motion for which neither of these functions is constant is called a precession of general form [1]. It is this case that we shall consider in the present paper. We attach a moving system of coordinates to the body in such a way that $\mathbf{a}=(0,0,1)$. Then one can ensure that relations (1.5), (1.7) and $\boldsymbol{\nu}\cdot\boldsymbol{\nu}=1$ are satisfied by setting

$$\nu_1=a_0'\sin'\varphi,\quad \nu_2=a_0'\cos\varphi,\quad \nu_3=a_0,\quad a_0'=\sin\theta_0 \tag{1.8}$$

Taking (1.3) and (1.4) into account, we substitute $\boldsymbol{\omega}$ from (1.6) into the first equation in (1.1) and integrals (1.2):

$$\varphi^{\bullet\bullet}A\mathbf{a}+\psi^{\bullet\bullet}A\mathbf{v}+\varphi^{\bullet}\psi^{\bullet}[\mathrm{Tr}(A)(\mathbf{v}\times\mathbf{a})-2(A\mathbf{v}\times\mathbf{a})]-$$
$$-\varphi^{\bullet 2}(A\mathbf{a}\times\mathbf{a})-\psi^{\bullet 2}(A\mathbf{v}\times\mathbf{v})-\varphi^{\bullet}(\mathbf{a}\times B\mathbf{v}+\boldsymbol{\lambda}\times\mathbf{a})-\psi^{\bullet}(\mathbf{v}\times B\mathbf{v}+\boldsymbol{\lambda}\times\mathbf{v})=\mathbf{s}\times\mathbf{v}+\mathbf{v}\times C\mathbf{v}$$

$$\varphi^{\bullet 2}(A\mathbf{a}\cdot\mathbf{a})+2\varphi^{\bullet}\psi^{\bullet}(A\mathbf{a}\cdot\mathbf{v})+\psi^{\bullet 2}(A\mathbf{v}\cdot\mathbf{v})=2(E+\mathbf{s}\cdot\mathbf{v})-C\mathbf{v}\cdot\mathbf{v} \tag{1.9}$$

$$\varphi^{\bullet}(A\mathbf{a}\cdot\mathbf{v})+\psi^{\bullet}(A\mathbf{v}\cdot\mathbf{v})=k-\boldsymbol{\lambda}\cdot\mathbf{v}+{}^1/_2(B\mathbf{v}\cdot\mathbf{v})$$

We project both sides of the first equality in (1.9) onto $\mathbf{a}$, $\boldsymbol{\nu}$, and $\boldsymbol{\nu}\times\mathbf{a}$:

$$\varphi^{\bullet\bullet}(A\mathbf{a}\cdot\mathbf{a})+\psi^{\bullet\bullet}(A\mathbf{a}\cdot\mathbf{v})=\psi^{\bullet 2}[\mathbf{a}\cdot(A\mathbf{v}\times\mathbf{v})]+$$
$$+\psi^{\bullet}[\mathbf{a}\cdot(\boldsymbol{\lambda}\times\mathbf{v})+\mathbf{a}\cdot(\mathbf{v}\times B\mathbf{v})]+\mathbf{a}\cdot(\mathbf{s}\times\mathbf{v})+\mathbf{a}\cdot(\mathbf{v}\times C\mathbf{v})$$

$$\varphi^{\bullet\bullet}(A\mathbf{a}\cdot\mathbf{v})+\psi^{\bullet\bullet}(A\mathbf{v}\cdot\mathbf{v})=2\varphi^{\bullet}\psi^{\bullet}[\mathbf{v}\cdot(A\mathbf{v}\times\mathbf{a})]+$$
$$+\varphi^{\bullet 2}[\mathbf{v}\cdot(A\mathbf{a}\times\mathbf{a})]+\varphi^{\bullet}[\mathbf{v}\cdot(\boldsymbol{\lambda}\times\mathbf{a})+\mathbf{v}\cdot(\mathbf{a}\times B\mathbf{v})]$$

$$\varphi^{\bullet\bullet}[A\mathbf{a}(\mathbf{v}\times\mathbf{a})]+\psi^{\bullet\bullet}[A\mathbf{v}\cdot(\mathbf{v}\times\mathbf{a})]+\varphi^{\bullet}\psi^{\bullet}[\mathrm{Tr}(A)a_0'^2- \tag{1.10}$$
$$-2(A\mathbf{v}\cdot\mathbf{v})+2a_0(A\mathbf{a}\cdot\mathbf{v})]-\varphi^{\bullet 2}[(A\mathbf{a}\cdot\mathbf{v})-a_0(A\mathbf{a}\cdot\mathbf{a})]-$$
$$-\psi^{\bullet 2}[a_0(A\mathbf{v}\cdot\mathbf{v})-(A\mathbf{a}\cdot\mathbf{v})]+\varphi^{\bullet}[a_0(\boldsymbol{\lambda}\cdot\mathbf{a})-(\boldsymbol{\lambda}\cdot\mathbf{v})-$$
$$-a_0(B\mathbf{a}\cdot\mathbf{v})+(B\mathbf{v}\cdot\mathbf{v})]+\psi^{\bullet}[(\boldsymbol{\lambda}\cdot\mathbf{a})-a_0(\boldsymbol{\lambda}\cdot\mathbf{v})-(B\mathbf{v}\cdot\mathbf{a})+$$
$$+a_0(B\mathbf{v}\cdot\mathbf{v})]+(\mathbf{a}\cdot\mathbf{s})-a_0(\mathbf{s}\cdot\mathbf{v})+a_0(C\mathbf{v}\cdot\mathbf{v})-(C\mathbf{v}\cdot\mathbf{a})=0$$

The gist of the method for investigating motion with precession about the vertical axis consists of the following [1]. From the first two equations in (1.10) we find the second derivatives $\varphi^{\bullet\bullet}$ and $\psi^{\bullet\bullet}$, and we substitute them into the third equation of this system. As a result, we obtain an equation containing the first derivatives $\varphi^{\bullet}$ and $\psi^{\bullet}$. On the basis of integrals (1.9), we can eliminate $\varphi^{\bullet}$ and $\psi^{\bullet}$ in the latter equation. On substituting expressions (1.8), the equation obtained in this way yields an equation of the form $F(\varphi, \lambda_i, s_j, A_{ij}, B_{kl}, C_{mn})=0$. The requirement that this equation should be an identity in φ leads to conditions for the parameters, under which the body will undergo a motion with precession. It can be shown that the above-mentioned transformations have no singularities.

In the general case the problem of precession for Eqs (1.1) has not been solved and only partial results are known [1].

In the present paper we state the problem of the conditions for the existence of precession of general form with the following structure:

$$\varphi^{\bullet 2}=b_1+b_2\sin\varphi,\quad \psi^{\bullet}=b_3/\varphi^{\bullet} \tag{1.11}$$

i.e. $\psi^{\bullet}\varphi^{\bullet}=b_3$, where b_3 is a constant. Precession of general form in the classical problem, which corresponds to the case of integrability found earlier in [2], has this property. In [1] the uniqueness of this precession was shown in the case when $\mathbf{a}$ belongs to the principal plane of the ellipsoid of inertia.

2. CONDITIONS FOR THE EXISTENCE OF A SOLUTION

Substituting (1.11) into the integrals from (1.9), we get

$$
\begin{gathered}
[f_1(\varphi)(b_1+b_2\sin\varphi)+b_3f_2(\varphi)]^2-g_2^2(\varphi)(b_1+b_2\sin\varphi)=0 \\
h_2(\varphi)(b_1+b_2\sin\varphi)-A_{33}(b_1+b_2\sin\varphi)^2-2b_3f_1(\varphi)(b_1+b_2\sin\varphi)-b_3^2f_2(\varphi)=0
\end{gathered}
\tag{2.1}
$$

Here

$$
\begin{gathered}
f_1(\varphi)=(A\mathbf{a}\cdot\mathbf{v})=a_1\sin\varphi+a_2\cos\varphi+a_3, \\
f_2(\varphi)=(A\mathbf{v}\cdot\mathbf{v})=c_1\sin 2\varphi+c_2\cos 2\varphi+c_3\sin\varphi+c_4\cos\varphi+c_5 \\
g_2(\varphi)={}^1/_2(B\mathbf{v}\cdot\mathbf{v})-(\boldsymbol{\lambda}\cdot\mathbf{v})+k=b_1^*\sin 2\varphi+b_2^*\cos 2\varphi+b_3^*\sin\varphi+b_4^*\cos\varphi+b_5^*, \\
h_2(\varphi)=2E+2(\mathbf{s}\cdot\mathbf{v})-(C\mathbf{v}\cdot\mathbf{v})=d_1\sin 2\varphi+d_2\cos 2\varphi+d_3\sin\varphi+d_4\cos\varphi+d_5 \\
a_1=A_{13}a_0', \quad a_2=A_{23}a_0', \quad a_3=A_{33}a_0, \quad c_1=A_{12}a_0'^2 \\
c_2={}^1/_2(A_{22}-A_{11})a_0'^2, \quad c_3=2A_{13}a_0a_0', \quad c_4=2A_{23}a_0a_0' \\
c_5={}^1/_2a_0'^2(A_{11}+A_{22})+A_{33}a_0^2 \\
b_1^*={}^1/_2B_{12}a_0'^2, \quad b_2^*={}^1/_4a_0'^2(B_{22}-B_{11}), \quad b_3^*=a_0'(B_{13}a_0-\lambda_1) \\
b_4^*=a_0'(B_{23}a_0-\lambda_2), \quad b_5^*={}^1/_4a_0'^2(B_{11}+B_{22})+{}^1/_2B_{33}a_0^2-\lambda_3a_0+k \\
d_1=-C_{12}a_0'^2, \quad d_2={}^1/_2(C_{11}-C_{22})a_0'^2, \quad d_3=2a_0'(s_1-C_{13}a_0) \\
d_4=2a_0'(s_2-C_{23}a_0), \quad d_5=2E+2s_3a_0-{}^1/_2a_0'^2(C_{11}+C_{22})-a_0^2C_{33}
\end{gathered}
$$

Equations (1.10) can also be written in a similar form. It is convenient to analyse the resulting relations starting from Eqs (2.1). As a result of the investigation carried out, the following conditions for the parameters are obtained:

$$
\begin{gathered}
A_{12}=A_{23}=0, \quad C_{12}=C_{13}=C_{23}=0 \\
C_{11}=C_{22}, \quad B_{12}=B_{23}=0, \quad B_{11}=B_{22} \\
4A_{13}^4+A_{13}^2(A_{11}-A_{22})(A_{11}+3A_{22}-4A_{33})-A_{11}A_{33}(A_{11}-A_{22})^2=0 \\
\lambda_1=B_{13}a_0, \quad \lambda_2=0, \quad \lambda_3=B_{11}b_1b_3^{-1}+a_0(B_{33}-B_{11}) \\
\operatorname{ctg}^2\theta_0=A_{22}A_{13}^2A_{33}^{-1}\zeta^{-1}, \quad \zeta=A_{13}^2-A_{33}(A_{11}-A_{22}) \\
s_1={}^1/_2b_3\zeta A_{13}^{-1}, \quad s_2=0, \quad s_3=a_0(C_{33}-C_{11})+ \\
+{}^1/_2b_3[A_{33}(A_{22}-A_{11})(A_{22}-A_{11}-2A_{33})- \\
-A_{13}^2(A_{11}+A_{22}+2A_{33})](A_{13}^2+\zeta)^{-1} \\
B_{11}=a_0'B_{13}b_3b_2^{-1}, \quad b_1=-a_0b_3(A_{13}^2+\zeta)A_{13}^{-2} \\
b_2=a_0'(A_{22}-A_{11})A_{13}^{-1}b_3
\end{gathered}
\tag{2.2}
$$

Thus, under conditions (2.2), Eqs (1.1) have the solution

$$
\begin{gathered}
\boldsymbol{\omega}=\varphi^{\cdot}\mathbf{a}+\psi^{\cdot}\mathbf{v}, \quad \mathbf{v}=(a_0'\sin\varphi,\ a_0'\cos\varphi,\ a_0) \\
\varphi^{\cdot}=(b_1+b_2\sin\varphi)^{1/2}, \quad \psi^{\cdot}=b_3/\varphi^{\cdot}
\end{gathered}
\tag{2.3}
$$

Let us mention the basic properties of the precession (2.3). Using (2.2) and the results obtained earlier in [1], we conclude that, as in the classical problem, in the given solution (2.3) the gyrostat is a Hess gyroscope (the centre of mass lies in the principal plane on the line perpendicular to the circular cross-section of the gyration ellipsoid), the vectors **s** and **a** lie in the principal plane of the ellipsoid of inertia, and θ_0 depends only on the moments of inertia of the gyrostat. If the case $B_{11} = B_{13} = B_{33} = 0$ is considered, then $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Since $C_{ij} = \varepsilon^2 A_{ij}$ (where ε^2 is a parameter) for the problem of the motion of a rigid body in a central Newtonian field of force, it follows by (2.2) that $A_{11} = A_{22}$, $A_{13} = 0$, and (2.3) is not a solution.

Under the given conditions (2.2), the axial moments of inertia and the components C_{11} and C_{33} of C and B_{13}

of B, as well as b_3 can be taken as the free parameters. That the solution (2.3) is real valued follows from the results obtained earlier in [1]. The dependence of φ on time can then be found by inverting an elliptic integral.

The method of investigation used in this paper indicates that the precession (2.3) is unique under the conditions (2.2). For $C = 0$, $B = 0$ and $\lambda = 0$, we obtain a precession of general form in the classical problem of the motion of a rigid body corresponding to a solution [2], which, despite the Hess conditions for the distribution of the mass of the body, does not occur as a special case in the Hess solution.

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THE EFFECT OF THE SURFACE TENSION GRADIENT ON THE MOTION OF A SPHERICAL AND DEFORMED DROP†

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It is shown that, when the Stokes equations are used, a drop which is falling through a viscous fluid can only maintain a strictly spherical shape when there are specific distributions of the surface tension. Deviations from these distributions will cause some deformation of the drop. These results are obtained using a more general solution of the Stokes equations compared with the solutions which were considered earlier [1].

THE MOTION of a spherical drop in a viscous fluid has been studied both theoretically and experimentally. It has been pointed out [2] that agreement between the experimental and theoretical results can be attained if account is taken of the effect of surfactants and the changes on the surface of a drop associated with them. Moreover, the surface tension distribution on the drop may manifest itself in the shape of its surface.

The deformation of a drop which falls through a viscous fluid has been treated in the Oseen approximation, taking into account inertial effects, by the method of matched asymptotic expansions [1, 3]. It was concluded [1] that deformations of the surface cannot occur and the drop will remain spherical within the framework of the inertia-less Stokes equations when the surface tension on the surface of the drop is constant and there is no change in the rate of flow around the spherical drop.

Let us consider the flow around a drop of radius R by another fluid with a velocity $\mathbf{U}$ at a large distance from the drop. This flow relative to the drop arises as a result of its falling through the fluid under the action of gravitational and Archimedean forces. The surface tension σ varies along the surface of the drop $\sigma(\theta)$. There are various reasons for this change in the surface tension: the existence of surfactants, a non-uniform temperature field, etc.

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